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APPLICATION OF THE BUBNOV-GALERKIN PROCEDURE TO THE PROBLEM OF SEARCHING FOR SELFOSCILLATIONS

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We propose the use of the Bubnov-Galerkin procedure to the search for self-oscillations. We establish the existence and the convergence of the approximations. In the basic case we have obtained the asymptotics of the rate of convergence. In [1] it was shown, on the basis of the results in [2], how we can construct finite-dimensional approximations to the periodic solutions of autonomous systems. Below we have pointed out another approach to solving the approximation problem, based on the parameter functionalization method proposed in [3].

1. We first consider an autonomous system of ordinary differential equations

$$dx/dt = f(x) \qquad (x \in R^n) \tag{1.1}$$

where f is a continuously differentiable mapping of a region $G \subset R^n$ into R^n. We assume that in region G system (1.1) has an isolated cycle Γ whose smallest positive period is ω_0. Let $x_0 \in \Gamma$ and let $x^*(t)$ be the solution of system (1.1) with the initial condition x_0 at $t = 0$. We assume cycle Γ to be simple, i.e. unity is a simple eigenvalue of the translation operator at time ω_0 along the trajectories of the variational system

$$d\xi/dt = f_x\,[x^*(t)]\,\xi \qquad (1.2)$$

Let $C\,[0, 1]$ be the Banach space of functions continuous on [0, 1] with values in R^n, assuming equal values at the endpoints of the interval [0, 1]. By P_m we denote the finite-dimensional projector associating with every continuous function $u(\tau) \in C\,[0, 1]$ having the Fourier series

$$u(\tau) \cong a_0 + \sum_{k=1}^{\infty} (a_k \cos 2k\pi\tau + b_k \sin 2k\pi\tau)$$

a part

$$u_m(\tau) = a_0 + \sum_{k=1}^{m} (a_k \cos 2k\pi\tau + b_k \sin 2k\pi\tau)$$

of the series. Let G_0 be some neighborhood of the element $u_0(\tau) = x^*(\tau\omega_0)$ $(0 \leqslant \tau \leqslant 1)$ of space $C\,[0,1]$. We assume that a strictly positive functional $\Omega(u)$ has been defined in G_0. The trigonometrical polynomials $u_m(\tau)$ $(0 \leqslant \tau \leqslant 1)$ which are the solutions of the finite-dimensional algebraic system

$$du_m/d\tau = P_m\Omega(u_m)\, f(u_m) \qquad (1.3)$$

are called the Galerkin approximations of system (1.1).

Theorem 1. Assume that functional $\Omega(u)$ is continuously differentiable at point u_0 and satisfies the conditions

$$\Omega(u_0) = \omega_0, \qquad \Omega_u(u_0)(du_0/d\tau) \neq 0$$

Then the Galerkin approximations u_m exist for sufficiently large m and converge to u_0; moreover, the following bounds on the rate of convergence

$$a_1 \,\|(I - P_m)\,u_0\|_C \leqslant \|u_0 - u_m\|_C \leqslant a_2\,\|(I - P_m)\,u_0\|_C$$

are valid for some $a_1, a_2 > 0$.

Proof. By the equality

$$Hu = \sum_{k=1}^{\infty} (2\pi k)^{-1} (-b_k \cos 2k\pi\tau + a_k \sin 2k\pi\tau)$$

we define an operator acting from the space $L_2\,[0, 1]$ of square-summable functions with values in R^n into the space $C\,[0, 1]$. Obviously, H is completely continuous. Using H we now introduce a completely continuous operator acting in space $C\,[0, 1]$ (analogous to the integral operator considered earlier in [4] for nonautonomous systems)

$$\mathbf{U}_\Omega u(\tau) = \int_0^1 \{u(\tau) + \Omega(u) f[u(\tilde{\tau})]\}\, d\tau + H(I - P_0)\,\Omega(u) f[u(\tau)] \qquad (u \in G_0) \qquad (1.4)$$

Let $u(\tau)$ be a fixed point of operator $\mathbf{U}_\Omega$, i.e.

$$u(\tau) \equiv \int_0^1 \{u(\tau) + \Omega(u) f[u(\tau)]\}\, d\tau + H(I - P_0)\,\Omega(u) f[u(\tau)] \qquad (1.5)$$

The function $u(\tau)$ assumes equal values at the endpoints of interval [0, 1]. Further, by integrating identity (1.5) over the interval [0, 1], we obtain

$$\int_0^1 \Omega(u) f[u(\tau)]\, d\tau = 0$$

Hence it follows that $f[u(\tau)] \equiv (I - P_0) f[u(\tau)]$. Therefore, we obtain $u'(\tau) \equiv \Omega(u) f[u(\tau)]$ by differentiating (1.4). Thus, the fixed points of operator $\mathbf{U}_\Omega$ are singly-periodic solutions of the system

$$du/d\tau = \Omega(u) f[u(\tau)] \tag{1.6}$$

The converse is true also. It is not difficult to see that the system of algebraic equations

$$u_m = P_m \mathbf{U}_\Omega u_m$$

is equivalent to system (1.3). Consequently, the question of Galerkin approximations to system (1.1) is equivalent to the question of searching for the usual Galerkin approximations for the equation

$$u = \mathbf{U}_\Omega u$$

in the Banach space C [0,1].

Let us show first of all that unity is not an eigenvalue of operator $(\mathbf{U}_\Omega)_u (u_0)$. To do this we write the equation $h = (\mathbf{U}_\Omega)_u (u_0)\, h$ in greater detail

$$h(\tau) = \int_0^1 h(\tau)\, d\tau + \int_0^1 \{\Omega(u_0) f_u [u_0(\tau)]\, h + \Omega_u(u_0) h f[u_0(\tau)]\}\, d\tau +$$
$$H(I - P_0)\{\Omega(u_0) f_u [u_0(\tau)]\, h + \Omega_u(u_0) h f[u_0(\tau)]\}$$

We see that $h(\tau)$ is a singly-periodic solution of the system of equations

$$dh/d\tau = \omega_0 f_u [u_0(\tau)]\, h + \Omega_u(u_0)\, h\, f[u_0(\tau)]$$

We set $\tau\omega_0 = t$, $\psi(t) = h(t/\omega_0)$. Then the function $\psi(t)$ is an ω_0-periodic solution of the system of equations

$$d\psi/dt = f_u[x^*(t)]\, \psi + \frac{1}{\omega_0} \Omega_u(u_0) h x^{\cdot *}(t)$$

The latter is possible only if

$$\Omega(u_0) h \int_0^{\omega_0} (x^{\cdot *}(t), \varphi(t))\, dt = 0$$

Here $\varphi(t)$ is an ω_0-periodic solution of the adjoint system

$$dy/dt = -f_u^* [x^*(t)]\, y$$

We consider two possible cases. At first let

$$\int_0^{\omega_0} (x^{\cdot *}(t), \varphi(t))\, dt = 0$$

Then, obviously,

$$(x^{\cdot *}(0), \varphi(0)) = 0 \tag{1.7}$$

By $\Phi(t)$ we denote the fundamental matrix of the system of differential equations (1.2) satisfying the condition $\Phi(0) = I$. We see that $\Phi(\omega_0)\, x^{\cdot *}(0) = x^{\cdot *}(0)$ and $\varphi(0) = \Phi^*(\omega_0)\, \varphi(0)$. Therefore, from equality (1.7) it follows that the equation

$$z = \Phi(\omega_0) z + x^{\cdot *}(0) \qquad (z \in R^n)$$

has a nontrivial solution. This contradicts the simplicity of the unit eigenvalue of the translation operator $\Phi(\omega_0)$.

If $\Omega_u(u_0)h = 0$, then, obviously, $\psi(t) = kx^*(t)$ and $h(\tau) = ku_0(\tau)$. Then from the hypotheses of Theorem 1 it follows that $k = 0$ and, therefore, $h(\tau) \equiv 0$. Consequently, unity is not an eigenvalue of operator $(\mathbf{U}_\Omega)_u(u_0)$. We note, finally, that operator $\mathbf{U}_\Omega$ can be represented as

$$\mathbf{U}_\Omega u = TSu$$

$$Tu = H(I - P_0)u + \int_0^1 u(\tau)\,d\tau$$

$$Su = \int_0^1 u(\tau)\,d\tau + \Omega(u) f[u(\tau)]$$

moreover,

$$P_m T = TP_m, \qquad \lim_{m\to\infty} \| T(I - P_m)\|_{L_2 \to C} = 0$$

The theorem's assertion now follows from a lemma established in [4]. The theorem is proved.

We should add that the existence and the convergence of the Galerkin approximations can be established even when system (1.1) has a family of cycles. For example, if the rotation $\gamma(I - \mathbf{U}_\Omega, G^\cdot)$ of a completely continuous vector field $I - \mathbf{U}_\Omega$ on the boundary $G^\cdot$ of some region G is nonzero, then for sufficiently large m the Galerkin approximations exist and converge to the set of singly-periodic solutions of system (1.6). The contiguity theorem established in [5] can prove useful for computing $\gamma(I - \mathbf{U}_\Omega, G^\cdot)$

2. Let us now consider a system of differential-difference equations

$$dx/dt = f[x(t - h_1), \ldots, x(t - h_k)] \tag{2.1}$$
$$x, f \in R^n, \qquad 0 \leqslant h_1 \leqslant \ldots \leqslant h_k$$

We assume that $f(x_1, \ldots, x_k)$ is defined and continuously differentiable in $R^n \times \ldots \times R^n$ and assumes values in R^n. Further, let there be an isolated cycle Γ in system (2.1), which is defined by the ω_0-periodic solution $x^*(t)$. We take it that the system of variational equations

$$\frac{d\xi}{dt} = \sum_{i=1}^{k} f_{x_i}[x^*(t - h_1), \ldots, x^*(t - h_k)]\,\xi \tag{2.2}$$

has a one-dimensional subspace of ω_0-periodic solutions. (Obviously, $x^{\cdot *}(t)$ belongs to this subspace). Finally, let the inequality

$$\int_0^{\omega_0} \left([x^{\cdot *}(t) + \sum_{i=1}^{k} h_i f_{x_i} x^{\cdot *}(t - h_i)],\ \varphi(t)\right) dt \neq 0$$

be valid for some ω_0-periodic solution $\varphi(t)$ of the system adjoint to (2.2) [6]. We say that such cycles are quasi-simple.

Let us show how the Galerkin procedure can be used for the approximate search for quasi-simple cycles. For this purpose we introduce the space $C_1[0,1]$ of functions $u(\tau)$ continuously differentiable on [0, 1] with values in R^n, for which $u(0) = u(1)$, $u^\cdot(0) = u^\cdot(1)$. The function $u_0(\tau) = x^*(\tau\omega_0)$ $(0 \leqslant \tau \leqslant 1)$ belongs to $C_1[0,1]$. We assume that a randomly-taken positive functional $\Omega(u)$ has been defined in some neighborhood G_0 of a point $u_0 \in C_1[0, 1]$, is continuously differentiable at point u_0

and satisfies the conditions

$$\Omega(u_0) = \omega_0, \ \Omega_n(u_0)(du_0 / d\tau) \neq 0$$

The trigonometrical polynomials u_m which are solutions of the following algebraic system

$$\frac{du_m}{d\tau} = P_m \Omega(u_m) f\left[u_m\left(\tau - \frac{h_1}{\Omega(u_m)}\right), \ldots, u_m\left(\tau - \frac{h_k}{\Omega(u_m)}\right)\right]$$

are called the Galerkin approximations to system (2.1).

Theorem 2. The Galerkin approximations exist for sufficiently large m and converge to u_0. Furthermore, the inequalities

$$a_1 \|(I - P_m) u_0\|_{C_1} \leqslant \|u_0 - u_m\|_{C_1} \leqslant a_2 \|(I - P_m) u_0\|_{C_1} \tag{2.3}$$

are valid for some $a_1, a_2 > 0$.

To prove this theorem we need to examine the equation $u = U_\Omega u$ in the space C_1 [0, 1] and to verify that: (1) unity is not an eigenvalue of the operator $(U_\Omega)_u (u_0)$; (2) $P_m T = T P_m$; (3) $\lim_{m \to \infty} \| T(I - P_m)\|_{L_2 \to C_1} = 0$. It remains vague whether estimates of type (2.3) can be established in the space C [0, 1] for differential-difference equations.

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